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On the stability of plane Couette flow to infinitesimal disturbances

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It has been conjectured for many years that plane Couette flow is stable to infinitesimal disturbances although this has never been proved. In this paper we use a combination of asymptotic analysis and numerical computation to examine the associated Orr–Sommerfeld differential problem in a systematic manner. We obtain new evidence that the conjecture is, in all probability, correct. In particular we find that, at a *fixed* large value of the Reynolds number R, *as in an experiment*, if a disturbance of wavenumber α has a damping rate $-\alpha c_i$ then $-c_i$ has a minimum value of order $R^{-\frac{1}{2}}$ when α is of order $R^{\frac{1}{2}}$. We believe that this result may be an essential prerequisite to an understanding of the stability of plane Couette flow to finite-amplitude disturbances.

1. Introduction

Throughout this paper our principal concern is with the stability of plane Couette flow to disturbances which are so small, 'infinitesimal', that we may ignore nonlinear effects. This classical problem has been the topic of several papers. We quote from Lin (1955); "... no conclusive answer has yet been reached concerning this problem even today. All existing investigations tend to show that the flow is stable". Now plane Couette flow is one of the simplest non-trivial solutions of the Navier–Stokes equations and it is therefore rather surprising that this linear stability problem still poses unanswered questions. We feel that this is partly because in the past the problem has not been examined in sufficient detail. The eigenvalue c of the associated Orr–Sommerfeld problem is a function of two parameters, the wavenumber α and the Reynolds number R, and it is not easy to evolve a systematic coverage of the whole α, R plane. We feel that a logical way to approach the problem is to suppose that the Reynolds number R has some large fixed value and then to examine how the eigenvalue c, and in particular its imaginary part c_i, varies as a function of α. Almost all the existing literature examines how c varies when α is kept fixed and R becomes large and as a consequence only that region of the α, R plane in which $\alpha \ll R^{\frac{1}{2}}$ is examined. It is in the above spirit that we shall approach the problem, but let us first review the existing literature.

In two important papers Wasow (1953) and Grohne (1954) developed similar asymptotic theories to show that two-dimensional disturbances of a fixed wavenumber α are stable when the Reynolds number R is sufficiently large.

This result is not however of primary importance as we shall explain in the next section. The first notable numerical attack was made by Gallagher & Mercer (1962). They carried out an extensive numerical investigation of the basic two-dimensional Orr–Sommerfeld problem for moderate values of α and R such that αR was less than about 1000, and indeed found that the flow was stable for this parameter range. Their work was substantiated by Deardorff (1963), who calculated some higher eigenvalues and extended slightly the parameter range covered by Gallagher & Mercer (1962). In a later paper Gallagher & Mercer (1964) also calculated some higher eigenvalues and they found that a mode-crossing phenomenon calculated by Grohne (1954) was erroneous. When R is fixed and α is so small that αR is less than about 75 the wave propagation speed is zero and Dikii (1964) has proved that in this event, when the eigenvalue c is purely imaginary, then the flow is stable. In another relevant paper Joseph (1968) used isoperimetric theory on the Orr–Sommerfeld equation for a general velocity profile $U(y)$ to show, by means of some elegant inequalities, that the flow is stable when either $R < 45{\cdot}6$ or $R^{-\frac{1}{2}}\alpha > 2^{-\frac{3}{4}}$; we shall find the latter result to be of particular interest. In an experiment Reichardt (1956) was able to maintain laminar flow for values of the Reynolds number up to about 750; transition to turbulence was evident at higher values and was presumably due to nonlinear effects.

Our principal objectives are to obtain new evidence that plane Couette flow is stable to infinitesimal disturbances and to find, when R is large, the value of α for which $-c_i$ is a minimum because wavenumbers close to this value may play a vital role in the nonlinear stability theory. In § 2 we consider in detail the classical Orr–Sommerfeld problem for an unmodulated plane wave propagating in the direction of the basic flow, especially when the Reynolds number has a large fixed value as it would have in a laboratory experiment. Because the problem contains two parameters the asymptotic ideas which we present in § 2 are not in themselves sufficient. It is necessary to use numerical calculations both to check and also to reinforce the asymptotic ideas and the novel numerical method which we use is presented in § 3. The numerical results are important and those of most interest are contained in figure 1, where $-c_i$ is plotted against α for several different values of R. We conclude the paper with a short discussion in § 4.

2. The two-dimensional Orr–Sommerfeld problem

We consider the flow of a viscous incompressible fluid between two horizontal plane walls which are at a distance $2L$ apart and which move so that the speed of the upper plane is U and of the lower plane is $-U$ in the same direction. We choose L, U and L/U as characteristic scales of length, speed and time respectively, with respect to which we make all our quantities non-dimensional. We use non-dimensional Cartesian co-ordinates (x, y, z) where the x axis lies between the planes and is in the streamwise direction, the z axis is the upward normal to the planes and the y axis is in the spanwise direction to form a right-handed system. Thus, the basic flow is in the positive-x direction and the two planes are

$z = \pm 1$; the pressure gradient is zero. Let the fluid have kinematic viscosity ν, then we define a Reynolds number by

$$R = UL/\nu. \tag{1}$$

Let $(u, 0, w)$ be the non-dimensional perturbation velocity components of a two-dimensional disturbance and let ϕ be the perturbation stream function so that

$$u = \partial\phi/\partial z, \quad w = -\partial\phi/\partial x. \tag{2}$$

If we substitute (2) with z added to u in the Navier–Stokes equations, neglect second-order quantities in ϕ, eliminate the pressure perturbation and seek a solution of the form

$$\phi = \psi(z)\exp\{i\alpha(x-ct)\}, \tag{3}$$

then we find that ψ satisfies the Orr–Sommerfeld equation

$$L\psi \equiv \{D^2-\alpha^2-i\alpha R(z-c)\}\{D^2-\alpha^2\}\psi = 0, \tag{4}$$

where $D \equiv d/dz$. The boundary conditions that the perturbation velocity components are zero at the plane walls become

$$\psi = D\psi = 0 \quad \text{at} \quad z = \pm 1. \tag{5}$$

The differential equation (4) and the boundary conditions (5) are homogeneous and so for given values of α and R a non-trivial solution for ψ exists only when $c = c_r + ic_i$ is a (complex) eigenvalue. The eigenvalues are discrete and form a complete set (DiPrima & Habetler 1969) and they may be ordered in decreasing magnitude of their imaginary part c_i. Moreover, Gallagher & Mercer (1964) found that this ordering was independent of the values of α and R, so that there is no 'mode-crossing', contrary to some calculations of Grohne (1954). For parallel flows the key role played by the Orr–Sommerfeld equation in linear stability theory may be rigorously justified by the use of Fourier–Laplace transform theory on the governing partial differential equation, but this justification restricts the wavenumber α to be real. No mathematical foundation exists at present for studies in which α is taken to be complex. Hence we suppose that α is a real positive wavenumber, so that the temporal damping rate of the disturbance is $-\alpha c_i$. If $c_i > 0$ the flow is unstable and if $c_i < 0$ then the flow is stable. In what follows we shall concentrate our attention on that eigenvalue which has the largest imaginary part since we wish to show that the flow is stable. We have obtained a considerable amount of information, both analytically and numerically, about the higher-mode eigenvalues but the results are of very little interest compared with those presented herein for the least stable mode.

Now c_i is a function of α and R and we wish to obtain some results, which, taken together with existing knowledge, indicate that c_i is almost certainly negative for all values of α and R. There are two main difficulties: first, because c_i is a function of *two* parameters our knowledge of the asymptotic dependence of c_i on α and R is necessarily limited to specific non-overlapping regions in the α, R plane; and, second, in those regions of the α, R plane which cannot be covered by asymptotic theory the larger α and/or R become the more difficult it is to calculate c_i accurately by a numerical process. Moreover, a *finite* number of numerical calculations of c_i for different values of α and R can never suffice to prove that $c_i < 0$ for *all* values of α and R. Gallagher & Mercer (1962) used a finite-difference matrix method to compute c_i for values of α up to 8 and R up to 500 and they found that c_i was negative for these ranges of the parameters. They also found, however, that $-c_i$ decreases without any discernible lower bound when R becomes very large. We know from experience that if a laminar flow is unstable so that it becomes turbulent then it is most likely to do so when the Reynolds number is large. So we focus attention on the situation when R is large and we shall use a combination of asymptotic analysis and numerical computation.

Although c_i is a function of both α and R the basic steady laminar flow knows only about R and is quite independent of α. Moreover, in an experiment R will usually have a large *fixed* value. Thus, we are firmly of the belief that a logical way to approach the problem is to determine how c_i varies as a function of α when R has some large fixed value. For a flow which is unstable to infinitesimal disturbances the value of α of most interest will be close to that value which makes $\beta_i = \alpha c_i$ a maximum (Stewartson & Stuart 1971). However, for a flow which is stable to infinitesimal disturbances, such as, we believe, plane Couette flow, then the value of α of most interest will be close to that value which makes $-c_i$ a minimum (Gill 1971; see appendix to Davey & Nguyen 1971) and so, in particular, we shall try to determine whether there is a value of α at which $-c_i$ is a minimum.

We begin with an asymptotic analysis similar to that used by Wasow (1953) and Grohne (1954), but which gives a much broader coverage of the α, R plane. For the presentation given below it is a pleasure to acknowledge my debt to Professor P. H. Roberts. Our hope is that the asymptotic analysis will provide us with an essential clue which will tell us what *structure* to seek in the numerical results so that we can make a good guess as to which quantities we should use for our ordinate and abscissa in figure 1.

We define λ, $\bar{c}$ and $f(z)$ by

$$\lambda^3 = i\alpha R \quad (\arg\lambda = \tfrac{1}{6}\pi), \tag{6}$$

$$\bar{c} = c + i\alpha/R, \tag{7}$$

$$f(z) = (D^2-\alpha^2)\,\psi(z), \tag{8}$$

so that we may write (4) as

$$D^2 f = \lambda^3(z-\bar{c})f. \tag{9}$$

The general solution for ψ from (8) which satisfies two of the boundary conditions (5), namely $\psi = D\psi = 0$ at $z = 1$, is

$$\psi = \frac{-1}{\alpha}\int_z^1 \sinh\alpha(z-s)f(s)\,ds. \tag{10}$$

The two remaining boundary conditions $\psi = D\psi = 0$ at $z = -1$ become, from (10),

$$\int_{-1}^1 \sinh\alpha(-1-s)f(s)\,ds = 0 \tag{11}$$

and

$$\int_{-1}^1 \cosh\alpha(-1-s)f(s)\,ds = 0. \tag{12}$$

Thus we have reduced the original problem to that of solving (9) subject to two integral conditions, which, by taking appropriate combinations of (11) and (12), may be written as

$$\int_{-1}^{1} \exp[\pm \alpha s] f(s)\, ds = 0. \tag{13}$$

But (9) is Airy's equation and so

$$f(z) = \frac{1}{2\pi i}\int_C \exp[\lambda(z-\bar{c})t - \tfrac{1}{3}t^3]\, dt, \tag{14}$$

where C is a suitably chosen contour in a complex-t plane as is usual for the Airy functions. So we may write the conditions (13) as

$$\int_{-1}^{1} \exp[\pm \alpha s] \int_C \exp[\lambda(s-\bar{c})t - \tfrac{1}{3}t^3]\, dt\, ds = 0, \tag{15}$$

and we may reverse the order of integration and integrate with respect to s to write (15) as

$$\int_C \frac{\sinh(\lambda t \pm \alpha)}{\lambda t \pm \alpha} \exp[-\lambda \bar{c} t - \tfrac{1}{3}t^3]\, dt = 0. \tag{16}$$

In the complex-t plane let C_1, C_2 and C_3 be contours from $\infty \exp[-\frac{2}{3}\pi i]$ to $\infty \exp[\frac{2}{3}\pi i]$, $+\infty$ to $\infty \exp[-\frac{2}{3}\pi i]$, and $\infty \exp[\frac{2}{3}\pi i]$ to $+\infty$ respectively and let the corresponding solutions of (9) and (14) be $A_1(\xi)$, $A_2(\xi)$ and $A_3(\xi)$, where $\xi \equiv \lambda(z-\bar{c})$; hence $A_1 + A_2 + A_3 = 0$, but any two of the functions are independent. For $-\frac{1}{3}\pi < \arg \xi < \frac{1}{3}\pi$, A_1 (or Ai in the usual notation) decreases exponentially like $\exp[-\frac{2}{3}\xi^{\frac{3}{2}}]$ as $|\xi| \to \infty$ whereas A_2 and A_3 increase exponentially. We follow Wasow (1953), see also Morawetz (1952), and assume that the disturbance is concentrated near the lower† plane wall so that the leading term in the asymptotic expansion for c is -1. Thus $f(z)$ must decrease rapidly away from $z = -1$ and we may take $f(z) \equiv \mathrm{Ai}(\xi)$, and $C = C_1$.

When $|\lambda| \gg 1$ so that αR is large there are two simple limiting cases which we may consider. First we consider the unfamiliar but important case in which $\alpha \gg |\lambda|$ also. Now provided that $|t| \ll \alpha/|\lambda|$, for the saddle-points over which C_1 passes in the integrals (17), we may replace the denominator in (16) by $\pm\alpha$, to reduce these conditions, at leading order, to

$$\int_{C_1} \exp[(\pm 1 - \bar{c})\lambda t - \tfrac{1}{3}t^3]\, dt = 0. \tag{17}$$

A justification for this assumption can be provided *a posteriori* from the results (18) and (19) given below. We shall discuss the error involved in accepting (17) later. If we compare (17) with (14) we see that the conditions (17) are simply $\mathrm{Ai}\{(\pm 1 - \bar{c})\lambda\} = 0$. The upper sign leads to no new information, since $(1-\bar{c})\lambda$ lies in the region where Ai is of exponentially small order. The lower sign requires

† The basic flow is an odd function of z so that by taking the complex conjugate of the Orr–Sommerfeld equation and then changing the sign of z it follows that if $c_r + ic_i$ is an eigenvalue then so is $-c_r + ic_i$, the associated disturbance is then concentrated near the *upper* plane wall.

that $-(1+\bar{c})\lambda$ coincides with one of the zeros of the Airy function, which are real. The most slowly decaying mode is obtained by selecting the largest of these, namely $-2{\cdot}3381\ldots$ (Miller 1946). The smaller zeros correspond to the higher modes which we mentioned earlier, and which are of much less interest. Thus, to leading order, we have shown that, for $\alpha \gg |\lambda| \gg 1$,

$$c_r = -1 + \frac{2{\cdot}3381}{(\alpha R)^{\frac{1}{3}}}\frac{\sqrt{3}}{2} \tag{18}$$

and

$$c_i = -\frac{\alpha}{R} - \frac{2{\cdot}3381}{(\alpha R)^{\frac{1}{3}}}\frac{1}{2}. \tag{19}$$

An alternative method is to note that for this case when α is large conditions (13) reduce to $f(1) = 0$ and $f(-1) = 0$. For the solution which is concentrated near $z = -1$ the first condition is satisfied automatically and the second condition is as given above. Physically $f(-1) = 0$ means that the disturbance vorticity is small at the wall compared with a typical value just away from the wall.

Second, we consider the familiar case when $|\lambda| \gg 1$ and also $|\lambda| \gg \alpha$ as studied by Wasow and Grohne. We may now write the lower-sign case of conditions (16), at leading order, as

$$\int_{C_1} t^{-1} \exp\left[(-1-\bar{c})\lambda t - \tfrac{1}{3}t^3\right] dt = 0; \tag{20}$$

the plus-sign condition gives no new information. We shall discuss the error involved in accepting (20) later. Moreover, (20) is just

$$\int_{-(1+\bar{c})\lambda}^{+\infty} \mathrm{Ai}(\xi)\, d\xi = 0. \tag{21}$$

We have used the numerical method described in §3 to calculate the value of $-(1+\bar{c})\lambda$ which satisfies (21) for the least stable mode directly from the Orr–Sommerfeld equation, and apart from the error term, for $|\lambda| \gg \alpha$ we find that

$$c_r = -1 + \frac{4{\cdot}1287}{(\alpha R)^{\frac{1}{3}}} \tag{22}$$

and

$$c_i = -\frac{\alpha}{R} - \frac{1{\cdot}0625}{(\alpha R)^{\frac{1}{3}}}. \tag{23}$$

Similar values for the coefficients of $(\alpha R)^{-\frac{1}{3}}$ in (22) and (23) were obtained by Zondek & Thomas (1953) in their study of a limiting case of plane Couette flow when one boundary is at infinity. Alternatively, we may note that when $|\lambda| \gg \alpha$ the exponential terms in (13) do not vary as rapidly as $f(s)$ grows or decays. Therefore both conditions (13) give

$$\int_{-1}^{1} f(s)\, ds = 0, \tag{24}$$

or, for the solution which is concentrated near $z = -1$,

$$\int_{-1}^{1} \mathrm{Ai}\{\lambda(z-\bar{c})\}\, dz = 0. \tag{25}$$

However, the upper limit in (25) may be replaced by ∞ since the integrand is of order $\exp[-\frac{2}{3}\xi^{\frac{3}{2}}]$ even when z is near 1, and so (25) is equivalent to (21). The essence of (24) physically is that there is only a little streamwise flux of disturbance vorticity.

Of the results obtained above the two of principal interest are (19) and (23). A more detailed analytical investigation indicates that when R is fixed and $\alpha \gg R^{\frac{1}{2}}$ then the error involved in (19) is of order $\alpha^{-\frac{5}{3}}$ and when α is fixed and $R^{\frac{1}{2}} \gg \alpha$ the error involved in (23) is of order $R^{-\frac{2}{3}}$. Moreover these results have been very carefully checked by a detailed numerical investigation of (4) and (5) for values of α up to 100 and values of αR up to 100 000 by use of the method described in § 3. So when R is large and α is not too small we may summarize the situation as follows: let $X = R^{-\frac{1}{2}}\alpha$ and $Y = -R^{\frac{1}{2}}c_i$ then for X large

$$Y - X = \Gamma_1 X^{-\frac{1}{3}} + O(X^{-\frac{5}{3}}), \tag{26}$$

and for X small

$$Y - X = \Gamma_0 X^{-\frac{1}{3}} + O(X^{\frac{1}{3}}), \tag{27}$$

where $\Gamma_0 = 1{\cdot}0625$ and $\Gamma_1 = 1{\cdot}1691$. When α and $R^{\frac{1}{2}}$ are of the same order of magnitude, so that X is neither large nor small, it is then difficult to make use of the integral conditions (16). In this case all the terms in the Orr–Sommerfeld equation (4) are of the same order of magnitude in the critical layer of the disturbance. To cover this parameter range we must embark upon a little speculation and we choose this in such a way that it may be either verified or found wanting by a numerical check.

We return to our original question and ask whether, when R is large, there is a value of α such that $-c_i$ is a minimum. Now the similarity in form of (26) and (27) and the closeness of the constants Γ_0 and Γ_1 suggests the possibility that for large values of R and α not too small then

$$Y = X + \Gamma(X)\, X^{-\frac{1}{3}}, \tag{28}$$

where $\Gamma(X)$ is a *slowly varying* function of X whose value remains close to Γ_0 and Γ_1. We now use (28), albeit tentatively, to obtain a clue which will tell us what structure to look for in the numerical results. If we keep R fixed and differentiate (28) with respect to α, ignoring the slow variation of $\Gamma(X)$ as α varies, and put $\partial c_i/\partial \alpha = 0$ we find that $-c_i$ has a minimum when α is of order $R^{\frac{1}{2}}$. To test this idea numerically we take several large fixed values of R and, for each value separately, compute c directly from (4) and (5) for a wide range of values of α and plot $Y = -R^{\frac{1}{2}}c_i$ against $X = R^{-\frac{1}{2}}\alpha$.

Some of the results which we have obtained are as shown in figure 1. The most important feature is that for values of $R \geqslant 200$ (and α not too small) all the results collapse onto the same line, which is shown as an unbroken curve in figure 1. The position of this curve was checked for values of α and αR as large as 100 and 100 000 respectively. It is just this feature which justifies our conjectural use of (28), for if $-c_i$ had been a minimum when α was proportional to some power of R other than $R^{\frac{1}{2}}$ the results for $R \geqslant 200$ would not lie on the same line. From this line we may readily calculate $\Gamma(X)$ and verify that it does vary slowly: as X increases from zero $\Gamma(X)$ increases from Γ_0 and attains a maximum value of $1{\cdot}38$ near $X = 0{\cdot}3$; as X becomes larger $\Gamma(X)$ decreases and approaches Γ_1 from

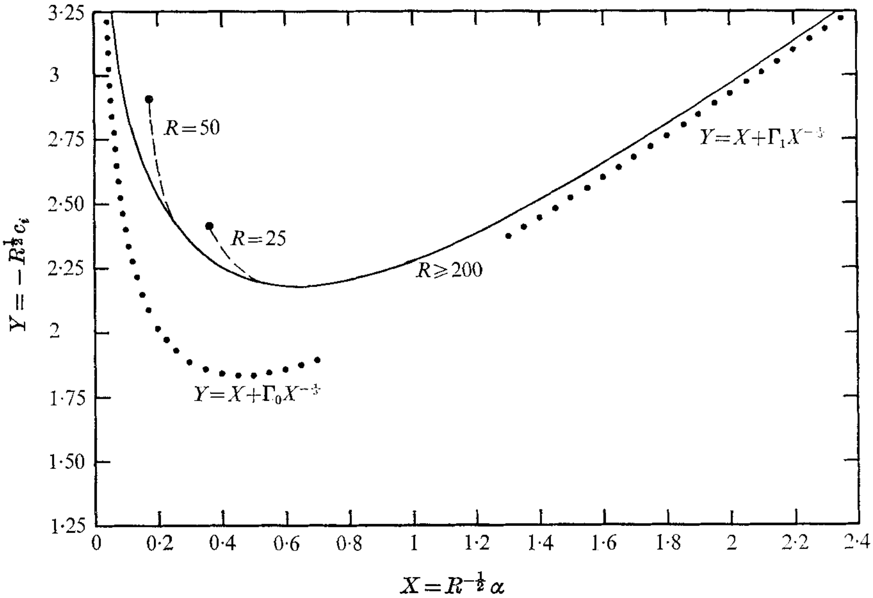

FIGURE 1. Plot of $Y = -R^{\frac{1}{3}} c_i$ against $X = R^{-\frac{1}{3}}\alpha$; the unbroken line may be used for $R \geqslant 200$. The point $X = 0{\cdot}6321$, $Y = 2{\cdot}1772$ gives the minimum value of $-c_i$ for $R \geqslant 200$. For comparison the curves for $R = 25$ and $R = 50$ are shown by broken lines; to the left of the dots on these curves $\alpha < \alpha_0$ and the disturbance takes the form of a standing wave. The dotted curves represent the leading terms of the asymptotic solutions with $\Gamma_0 = 1{\cdot}0625$ and $\Gamma_1 = 1{\cdot}1691$.

above. For $R \geqslant 200$, $-c_i$ is a minimum when $\alpha = 0{\cdot}6321\, R^{\frac{1}{3}}$, at which value of α $-c_i = 2{\cdot}1772 R^{-\frac{1}{3}}$; we note the proximity to the minimum of the result of Joseph (1968) that the flow is stable if $R^{-\frac{1}{3}}\alpha > 2^{-\frac{3}{4}} = 0{\cdot}5946$.

In figure 1 for all values of R the separate curves soon coalesce on the right-hand side of the minimum and when $\alpha \gg R^{\frac{1}{3}}$ then $-c_i$ increases as α becomes larger and approaches the value α/R from above. On the left-hand side of the minimum the smaller values of R peel off upwards as α decreases. When α becomes so small that αR is no longer large $-c_i$ continues to increase until the disturbance takes the form of a standing wave. This occurs when $\alpha < \alpha_0$, where α_0 is approximately $75{\cdot}48R^{-1}$ (see figure 2 of Gallagher & Mercer 1962); at $\alpha = \alpha_0$ there is a cusp. When $\alpha < \alpha_0$ then $-c_i$ does have another minimum but this is larger than the minimum for $\alpha > \alpha_0$ when $R > 72.3$. Our numerical calculations indicate that $-\beta_i = -\alpha c_i$ always has its minimum value when α lies in the range $0 < \alpha < \alpha_0$, and for this range Dikii (1964) has proved that $-\beta_i > \alpha^2/R$; when α is very small $-\beta_i R$ tends to π^2. Values of $-c_i$ when $\alpha < \alpha_0$ are not shown in figure 1 and $\alpha = \alpha_0$ at the dots on the left-hand ends of the broken curves for $R = 25$ and 50. We have done calculations for a much wider range of values of α and R than those shown in figure 1 and we have also investigated the higher modes but the corresponding results are simply of no interest compared with those shown in figure 1. In fact figure 1 covers quite a wide range anyway for if, say, $R = 1000$ then the range of α covered is approximately $1{\cdot}7 < \alpha < 73$.

It is of interest to note that, at least in retrospect, the 'core' of the problem may be obtained without having to do any calculations for values of R larger than a few hundred. That the important values of α are probably of order $R^{\frac{1}{2}}$ might be expected from the following simple physical argument. If members of a 'set' of disturbances all have the same energy $E \sim u^2 + w^2$ then our knowledge of turbulence tells us that it will be just those members of the set with $u \sim w$ which will be most likely to tempt a laminar flow to become turbulent. The continuity equation indicates that if $u \sim w$ then $\partial/\partial x \sim \partial/\partial z$, but $\partial/\partial x \sim \alpha$ and in the critical layer $\partial/\partial z \sim (\alpha R)^{\frac{1}{3}}$, so that $\alpha \sim (\alpha R)^{\frac{1}{3}}$ and therefore $\alpha \sim R^{\frac{1}{2}}$.

3. The numerical method

The problem defined by (4) and (5) is difficult to solve numerically when either α or αR is large because then the characteristic values of the Orr–Sommerfeld differential operator L differ greatly in their real parts. The viscous complementary solutions dominate the inviscid ones and make it difficult to determine what linear combination of the individual solutions will satisfy the boundary conditions. We wish to calculate the principal eigenvalue c for as many different values of α and R as possible in the ranges $0 \leqslant \alpha \leqslant 100$ and $0 \leqslant \alpha R \leqslant 100000$, and to reduce the necessary computing time to a minimum with the proviso that a simple method may be used.

A powerful and efficient method available for solving two-point boundary-value problems is the parallel shooting procedure as developed by Keller (1968). This excellent method is a cross between matrix methods and marching or shooting methods; it combines the advantages of both methods and eliminates most of their disadvantages. A simple special case of Keller's procedure is the method of orthonormalization. We have used this method in a novel form of 'complete' orthonormalization, which is especially simple to use, to obtain not only the results shown in figure 1 but also many others in support of the content of this paper. We now give a brief,† but essentially complete, description of the method. (The description given in Davey & Nguyen (1971) is not complete and should be ignored.)

We shall use Runge–Kutta integration over the range $-1 \leqslant z < 1$ and choose say m steps of equal length $h = 2/m$. We assume that the required eigenvalue c is already known approximately, if not then it will be necessary to do a little preliminary work using a variational or matrix type of method as has been well explained by Lee & Reynolds (1967). Let $\mathbf{y} = \{\psi, D\psi, D^2\psi, D^3\psi\}$ so that for each value of z the corresponding element $\mathbf{y}$ belongs to a four-dimensional vector space. Now suppose that when $z = ih$ then $\mathbf{y} = \mathbf{y}_i$ and consider integration, using a specific routine, from $z = ih$ to $z = (i+1)h$. Then the value $\mathbf{y}_{i+1}$ of $\mathbf{y}$ at $z = (i+1)h$ will be given by

$$\mathbf{y}_{i+1} = \mathbf{A}^i \mathbf{y}_i, \tag{29}$$

where $\mathbf{A}^i$ is a 4×4 matrix whose elements will be independent of $\mathbf{y}_i$ because L is a linear operator; but they will depend upon the *current* value of c. We may

† For an exhaustive account of orthonormalization methods see Davey (1973).

determine $\mathbf{A}^i$ by letting $\mathbf{y}_i$ have separately the values $\{1, 0, 0, 0\}$, $\{0, 1, 0, 0\}$, $\{0, 0, 1, 0\}$ and $\{0, 0, 0, 1\}$ and integrating from $z = ih$ to $z = (i+1)h$. So for each of the m integration steps we may find the corresponding m *transfer* matrices $\mathbf{A}^i$. It is just because we find these transfer matrices as described above that our orthonormalization may be said to be *complete*.

The relationship between $\mathbf{y}$ at $z = -1$ and $\mathbf{y}$ at $z = 1$ is of the form

$$\mathbf{y}_{z=1} = \mathbf{B}\mathbf{y}_{z=-1}, \tag{30}$$

where $\mathbf{B}$ is the 4×4 matrix which is the left-product of all the transfer matrices $\mathbf{A}^i$. If the real parts of the characteristic values of L are not greatly different, so that R is not too large, then $\mathbf{B}$ may be found directly by straightforward shooting and the transfer matrices need not be found individually. The reason for this is that four integrations starting with four orthonormal values of $\mathbf{y}$ at $z = -1$ will be such that the four corresponding values of $\mathbf{y}_i$ at the end of *every* integration step will still be 'good' linearly independent vectors. By 'good' here we mean that the vectors shall still be nearly orthonormal. If, however, R is so large that the characteristic values of L are greatly different then round-off errors accompanied by the most rapidly growing solution will malform $\mathbf{B}$ and we shall not be able to calculate $\mathbf{B}$ with sufficient accuracy by straightforward shooting. What has happened is that, by the time we have integrated halfway across (say), if the four values are $\mathbf{B}^1\mathbf{y}_{z=-1}$, where $\mathbf{B} = \mathbf{B}^2\mathbf{B}^1$, then the vectors $\mathbf{B}^1\mathbf{y}_{z=-1}$ will be so nearly parallel that, computationally, they will form too poor a base from which to determine $\mathbf{B}^2$ accurately.

Instead we must proceed as follows: let the number of integration steps be $m = pq$, where p and q are both integers with q large and fixed and usually with p small although the larger α or R become the larger p, and so also m, must be made. As a general rule we may take say $q = 100$ and p a small multiple of the largest characteristic value, the actual multiple depending upon the accuracy required. (For plane Couette flow the characteristic values are of order α and $(\alpha R)^{\frac{1}{2}}$.) The first stage is to calculate the products in blocks of q of the transfer matrices $\mathbf{A}^i$ by integrating over blocks of q steps using orthonormal initial values; it is not necessary to find each $\mathbf{A}^i$ individually unless a fine tabulation of the eigenfunction is required. Let the p matrices so obtained be called $\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3, \ldots, \mathbf{B}^p$ such that $\mathbf{B}^1$ is the left-product of the first q matrices $\mathbf{A}^i$ and $\mathbf{B}^2$ is the left-product of the next q matrices $\mathbf{A}^i$ and so forth. Equivalently we may think of the range of integration as being divided into only p steps which may themselves be subdivided but only for the specific purpose of using the Runge–Kutta routine.

The second stage is to left-multiply the matrices $\mathbf{B}^j$, $1 \leqslant j \leqslant p$, together in a special way to obtain accurately some key information about the full transfer matrix

$$\mathbf{B} = \mathbf{B}^p\mathbf{B}^{p-1} \ldots \mathbf{B}^2\mathbf{B}^1. \tag{31}$$

Now the boundary conditions (5) tell us that the eigenvalue c must be iterated upon until the determinant of the upper right-hand quarter of $\mathbf{B}$ is zero. This is because the first two components of both $\mathbf{y}_{z=-1}$ and $\mathbf{y}_{z=1}$ are zero, so that equation (30) is of the form

$$\begin{pmatrix} 0 \\ 0 \\ ? \\ ? \end{pmatrix} = \left(\begin{array}{cc|cc} \times & \times & \bullet & \bullet \\ \times & \times & \bullet & \bullet \\ \hline \times & \times & \times & \times \\ \times & \times & \times & \times \end{array}\right) \begin{pmatrix} 0 \\ 0 \\ ? \\ ? \end{pmatrix}. \qquad (32)$$

So we calculate this determinant accurately as follows: we add a multiple of the third column of $\mathbf{B}^1$ to the fourth column so that these columns become orthgonal; we then normalize every column and left-multiply by $\mathbf{B}^2$. Let the new matrix so obtained be called $\mathbf{C}$. We now repeat this process on $\mathbf{C}$ so that the last two columns are made orthogonal in the same way and then every column is normalized. Next, we left-multiply this new $\mathbf{C}$ by $\mathbf{B}^3$ and repeat the process to form another $\mathbf{C}$ and so forth until the multiplication by $\mathbf{B}^p$ and a final use of the process have been completed. Thus, the final matrix $\mathbf{C}$ will be a modification of $\mathbf{B}$ such that its last two columns are orthonormal and such that the determinant, or rather the singularness, of the upper right-hand quarter is essentially unchanged. This part of the matrix is now well-formed and we may use any standard iterative technique to adjust c until this determinant is sufficiently close to zero; the matrices $\mathbf{B}^j$ must of course be recalculated each time that c is changed in the iteration procedure.

The above method may be readily adapted for other similar problems of higher differential order. For an nth-order differential system the main difference is that the matrices will be $n \times n$ instead of 4×4. The boundary conditions will determine how best to define the components of $\mathbf{y}$ so that part of $\mathbf{B}$ should have a zero determinant, and thus also which columns should be made orthogonal. It is usually only necessary for the number p of orthonormalizations to be very small; for (4) and (5) with α fixed the Reynolds number may be approximately quadrupled each time that p is doubled. What is essential is that the solid which may be defined by the column (or row) vectors of the transfer matrix between two successive orthonormalization points shall not differ markedly from a cube. The integration steps may each be of different length without introducing any complication, also the integration may be over any range and in either direction.

4. Concluding remarks

We have examined the principal eigenvalue c of the two-dimensional Orr–Sommerfeld problem for plane Couette flow over a wide range of values of the wavenumber α and the Reynolds number R. Specifically we have investigated how c varies with α when R has a large fixed value, and to do this we have used a delicate combination of asymptotic analysis and numerical computation. Our main aim has been to find the value of α at which $-c_i$ is a minimum because the neighbourhood of this wavenumber is probably important as regards the stability of plane Couette flow to small- but finite-amplitude disturbances in contrast to flows which are unstable to infinitesimal disturbances when the important values of α are those near which αc_i is a maximum. We have also obtained new evidence that plane Couette flow is stable to infinitesimal disturbances.

When $\alpha \gg R^{\frac{1}{2}}$ we have found that $-c_i$ is positive and increases as α becomes larger and $-c_i > \alpha/R$. When α is smaller but of order $R^{\frac{1}{2}}$ then $-c_i$ has its smallest value (provided $R > 72{\cdot}3$) of about $2{\cdot}2R^{-\frac{1}{2}}$ when α is about $0{\cdot}63R^{\frac{1}{2}}$. As α decreases so that $\alpha \ll R^{\frac{1}{2}}$ then $-c_i$ increases until α becomes so small that $c_r = 0$ and the disturbance takes the form of a standing wave.

We note that Squire's (1933) theorem is valid and so the propagation of an oblique wave poses the same eigenvalue problem as a two-dimensional wave at a lower value of the Reynolds number. Therefore $-c_i$ will be bounded below by a larger number than if the wave were two-dimensional because the minimum value of $-c_i$ is of order $R^{-\frac{1}{2}}$.

Ideas similar to those presented in this paper may be used to show that in the corresponding stability problem for flow in a circular pipe when R is large $-c_i$ has a minimum value of order $R^{-\frac{2}{3}}$ when the axial wavenumber is of order $R^{\frac{1}{3}}$ whether or not the disturbance is axisymmetric.

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